Closed Form Solution for vibration of Timoshenko beams with single discontinuity

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Abstract - Vibration functions of a Timoshenko beam with arbitrary discontinuities are derived. Heaviside's function is employed here to account for the discontinuity points in the beam so that the modal displacement and rotation can be described by a single function. Consequently the solution of vibration is significantly simplified. The application of present model to smart structure lead-zirconate-titanate (PZT) actuatorand damage detection are presented.

Index Terms - Vibration, Timoshenko beam, Euler-Bernoulli beam, finite element method, cantilever beam, multiple cracks

1. Introduction

Structure maintenances are gaining much more attention in the field of engineering because more and more old machines and buildings are eroded or damaged with time going. Inspection of these damages inside the structural components is playing the main role in judging how serious the damages are and whether it needs to be repaired immediately. A lot of efforts have been made to find methodsfor early detecting and locating the damages. Most of the damages are presented as cracks and the cracks can reduce the stiffness of structure, therefore the dynamics characteristics, such as natural frequency and mode shapes, related to structural stiffness will change correspondingly.

In the last three decades numerous damage detection methods based on dynamic response are developed [1], [2], [3], [4]. Diamarogonas [5] gave a review of the state of the art of vibration based methods to detect the cracks in the structures. Diamarogonas modeled the crack as a massless rotational spring and its equivalent stiffness was computed as a function of crack depth by employing fracture mechanics methods. Subsequently, many researchers build various molds and methods to determine the size and position of one crack in a beam [6],[7],[8],[9]. Later Ostachowicz and Krawczuck[10] studied on the dynamic behaviors of a beam with two cracks. Double crack beams are investigated intensively by analytical, numerical and experimental methods [11], [12]. Shifrin and Ruotolo [13] proposed a method for evaluating natural frequencies of such a beam that requires calculation of an "n+2" determination instead of a "4n+4" to study the natural frequencies of a beam with an arbitrary number of cracks. A more simplified method was proposed by Khiemand Lien [14] for evaluating the natural frequencies of beams with an arbitrary number of cracks.

All the above studies are concentrated on the analysis of the effect of transverse cracks. There are also other discontinuities such as intermediate resilient support, internal hinge or concentrated forces, etc. in the field of civil and mechanical engineering. Different discontinuities can complex the analysis of the vibration significantly. The approximated methods are employed to solve the high order eigenvalue equations [15], [16], [17]. Recently a general solution to account for arbitrary combinations of discontinuities and boundary conditions are proposed by Wang and Qiao[18]. A general function technique developed by Yavari [19] is employed here to achieve only one general displacement function to describe the whole beam. Heaviside and Dirac delta functions are used to account for various discontinuity conditions. But the beam model used in Wang and Qiao[18] paper is Euler-Bernoulli beam formulation, which do not consider the effects of shear deformation and rotary inertia. Especially, for short beams, such effects cannot be neglected.

In this paper the general solution of modal displacement of a Timoshenko beam with arbitrary discontinuities and boundary conditions is derived. The solution for the vibration of a Timoshenko beam is more complicated than that of Euler-Bernoulli beam [20]. To verify the accuracy of the model, two applications are presented and compared with Wang and Qiao[18]'s solution and finite element method results.

2.THERORETICAL

2.1 Vibration of Timoshenko beam with one discontinuity point

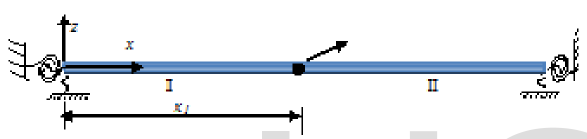

Fig.1. A Timoshenko beam with one discontinuity point at x1

As shown in Fig.1, a Timoshenko under general boundary conditions with one discontinuity at point x_1, which separate the whole beam into portion I and II. The discontinuity term at this point can be the deflection, slope, curvature, the third order derivative of the deflection, or any combination of above four terms. The equations of motion for each segment:

$$kGA\frac{\partial}{\partial x}\left(\psi_1(x,t)-\frac{\partial w_1(x,t)}{\partial x}\right)+\rho A\frac{\partial^2 w_1(x,t)}{\partial t^2}=0 \qquad (1)$$

$$kGA\frac{\partial}{\partial x}\left(\psi_2(x,t)-\frac{\partial w_2(x,t)}{\partial x}\right)+\rho A\frac{\partial^2 w_2(x,t)}{\partial t^2}=0 \qquad (2)$$

$$EI\frac{\partial^2 \psi_1(x,t)}{\partial x^2}-kGA\left(\psi_1(x,t)-\frac{\partial w_1(x,t)}{\partial x}\right)-\rho I\frac{\partial^2 \psi_1(x,t)}{\partial t^2}=0 \qquad (3)$$

$$EI\frac{\partial^2 \psi_2(x,t)}{\partial x^2}-kGA\left(\psi_2(x,t)-\frac{\partial w_2(x,t)}{\partial x}\right)-\rho I\frac{\partial^2 \psi_2(x,t)}{\partial t^2}=0 \qquad (4)$$

Where $w_1(x,t)$ and $w_2(x,t)$ are the transverse deflections of the segments I and II, respectively; $\psi_1(x,t)$ and $\psi_2(x,t)$ are the rotations of the segments I and II, respectively; E and I are the Young's modulus and moment inertia of the beam, respectively; G and k shear modulus and numerical factor depending on the shape of the cross section; ρ and A are the density and cross-section area of the beam, respectively.

The deflection and rotation of the whole beam are expressed in term of a single function, respectively.

$$\Delta w(x,t)=w_2(x,t)-w_1(x,t) \qquad (5)$$

$$w(x,t)=w_1(x,t)+\Delta w(x,t)H(x-x_1) \qquad (6)$$

$$\Delta\psi(x,t)=\psi_2(x,t)-\psi_1(x,t) \qquad (7)$$

$$\psi(x,t)=\psi_1(x,t)+\Delta\psi(x,t)H(x-x_1) \qquad (8)$$

Where $w(x,t)$ and $\psi(x,t)$ are the deflection and rotation functions of the whole beam, which are a generalized function with discontinuities at location x_1. $H(x-x_1)$ is the Heaviside function which jumps from zero to unit at location x_1.

Differentiating both sides of (5-8) with respect to x, we have

$$w'(x,t)=w_1'(x,t)+\Delta w'(x,t)H(x-x_1)+\Delta w(x_1,t)\delta(x-x_1) \qquad (9)$$

$$w''(x,t)=w_1''(x,t)+\Delta w''(x,t)H(x-x_1)+\Delta w'(x_1,t)\delta(x-x_1)+\Delta w(x_1,t)\delta'(x \qquad (10)$$

$$\psi'(x,t)=\psi_1'(x,t)+\Delta\psi'(x,t)H(x-x_1)+\Delta\psi(x_1,t)\delta(x-x_1) \qquad (11)$$

$$\psi''(x,t)=\psi_1''(x,t)+\Delta\psi''(x,t)H(x-x_1)+\Delta\psi'(x_1,t)\delta(x-x_1)+\Delta\psi(x_1,t)\delta'(x-x_1) \qquad (12)$$

where the prime dot over $w_i\ (i=1,2)$ is the derivative of the transverse deflection and rotation with respect to x. $\delta(x-x_1)$ is Dirac delta function.

Combine equation (1) and (2), (3) and (4):

$$\frac{\partial}{\partial x}\left(\psi_1(x,t)-\frac{\partial w_1(x,t)}{\partial x}\right)+\frac{\rho A}{B}\frac{\partial^2 w_1(x,t)}{\partial t^2}+\left[\begin{array}{l}\frac{\partial}{\partial x}\left(\psi_2(x,t)-\frac{\partial w_2(x,t)}{\partial x}\right)+\frac{\rho A}{B}\frac{\partial^2 w_2(x,t)}{\partial t^2}\\-\left(\frac{\partial}{\partial x}\left(\psi_1(x,t)-\frac{\partial w_1(x,t)}{\partial x}\right)+\frac{\rho A}{B}\frac{\partial^2 w_1(x,t)}{\partial t^2}\right)\end{array}\right]H(x-x_1)=0 \qquad (13)$$

$$\frac{\partial^2\psi_1(x,t)}{\partial x^2}-\frac{B}{D}\left(\psi_1(x,t)-\frac{\partial w_1(x,t)}{\partial x}\right)-\frac{\rho I}{D}\frac{\partial^2\psi_1(x,t)}{\partial t^2}+\left[\begin{array}{l}\frac{\partial^2\psi_2(x,t)}{\partial x^2}-\frac{B}{D}\left(\psi_2(x,t)-\frac{\partial w_2(x,t)}{\partial x}\right)-\frac{\rho I}{D}\frac{\partial^2\psi_2(x,t)}{\partial t^2}\\-\left(\frac{\partial^2\psi_1(x,t)}{\partial x^2}-\frac{B}{D}\left(\psi_1(x,t)-\frac{\partial w_1(x,t)}{\partial x}\right)-\frac{\rho I}{D}\frac{\partial^2\psi_1(x,t)}{\partial t^2}\right)\end{array}\right]H(x-x_1)=0 \qquad (14)$$

where $B=kGA$ and $D=EI$

Rearranging equations (13) and (14):

$$\frac{\partial \psi_1(x,t)}{\partial x}+\frac{\partial \Delta\psi(x,t)}{\partial x}H(x-x_1)-\left(\frac{\partial^2 w_1(x,t)}{\partial x^2}+\frac{\partial^2 \Delta w(x,t)}{\partial x^2}H(x-x_1)\right)$$
$$=-\frac{\rho A}{B}\left(\frac{\partial^2 w_1(x,t)}{\partial t^2}+\frac{\partial^2 \Delta w(x,t)}{\partial t^2}H(x-x_1)\right) \quad (15)$$

$$\frac{\partial^2\psi_1(x,t)}{\partial x^2}+\frac{\partial^2 \Delta\psi(x,t)}{\partial x^2}H(x-x_1)-\frac{B}{D}\left(\psi_1(x,t)+\Delta\psi(x,t)H(x-x_1)\right)$$
$$+\frac{B}{D}\left(\frac{\partial w_1(x,t)}{\partial x}+\frac{\partial \Delta w(x,t)}{\partial x}H(x-x_1)\right)$$
$$=\frac{\rho I}{D}\left(\frac{\partial^2\psi_1(x,t)}{\partial t^2}+\frac{\partial^2 \Delta\psi(x,t)}{\partial t^2}H(x-x_1)\right) \quad (16)$$

Substitute (5) – (12) to (15) and (16)

$$\psi'(x,t)-w''(x,t)+\frac{\rho A}{B}\frac{\partial^2 w}{\partial t^2}=\Delta\psi(x_1,t)\delta(x-x_1)$$
$$-\Delta w'(x_1,t)\delta(x-x_1)-\Delta w(x_1,t)\delta'(x-x_1) \quad (17)$$

$$\psi''(x,t)-\frac{B}{D}\psi(x,t)+\frac{B}{D}w'(x,t)-\frac{\rho I}{D}\frac{\partial^2\psi}{\partial t^2}(x,t)$$
$$=\Delta\psi'(x_1,t)\delta(x-x_1)+\Delta\psi(x_1,t)\delta'(x-x_1)+\frac{B}{D}\Delta w(x_1,t)\delta(x-x_1) \quad (18)$$

Equations (17) and (18) give the equations of motion of the Timoshenko beam with discontinuities in term of general equation $w(x,t)$ and $\psi(x,t)$ .Considering free vibration or harmonic forced vibration, (17) and (18) can be solved through variable separation method.

Let

$$w(x,t)=W(x)\sin(wt) \quad (19)$$
$$\psi(x,t)=\Psi(x)\sin(wt) \quad (20)$$

where $W(x)$ and $\Psi(x)$ are the modal displacement and rotation of the beam. Substituting (19) and (20) into (17) and (18), we have

$$\Psi'(x)-W''(x)-\frac{\rho A w^2}{B}W(x)=\Delta\Psi(x)\delta(x-x_1)$$
$$-\Delta W'(x_1)\delta(x-x_1)-\Delta W(x_1)\delta'(x-x_1) \quad (21)$$

$$\Psi''(x)-\frac{B}{D}\Psi(x)+\frac{B}{D}W'(x)+\frac{\rho I w^2}{D}\Psi(x)=\Delta\Psi'(x_1)\delta(x-x_1)$$
$$+\Delta\Psi(x_1)\delta'(x-x_1)+\frac{B}{D}\Delta W(x_1)\delta(x-x_1) \quad (22)$$

Let $\frac{\rho A w^2}{B}=\alpha^2$, $\frac{B}{D}=\beta_1^2$, $\frac{\rho I w^2}{D}=\beta_2^2$

Applying Laplace transform to (21) and (22)

$$s\Psi(s)-(s^2+\alpha^2)W(s)+sW(0)+W'(0)-\Psi(0)=$$
$$\Delta\Psi(x_1,t)e^{-sx_1}-s\Delta W(x_1)e^{-sx_1}-\Delta W'(x_1)e^{-sx_1} \quad (23)$$

$$(s^2-\beta_1^2+\beta_2^2)\Psi(s)+\beta_1^2 sW(s)-s\Psi(0)-\Psi'(0)-\beta_1^2W(0)$$
$$=\Delta\Psi'(x_1)e^{-sx_1}+s\Delta\Psi(x_1)e^{-sx_1}+\beta_1^2\Delta W(x_1)e^{-sx_1} \quad (24)$$

$W(s)$ and $\Psi(s)$ can be easily found by (23) and (24):

$$\left[\frac{\beta^2 s}{(s^2-\beta_1^2+\beta_2^2)}+\frac{(s^2+\alpha^2)}{s}\right]W(s)-\frac{s\Psi(0)+\Psi'(0)+\beta_1^2W(0)}{(s^2-\beta_1^2+\beta_2^2)}$$
$$-\frac{sW(0)+W'(0)-\Psi(0)}{s}=\frac{\Delta\Psi'(x_1)e^{-sx_1}+s\Delta\Psi(x_1)e^{-sx_1}+\beta_1^2\Delta W(x_1)e^{-sx_1}}{(s^2-\beta_1^2+\beta_2^2)}$$
$$-\frac{\Delta\Psi(x_1)e^{-sx_1}-s\Delta W(x_1)e^{-sx_1}-\Delta W'(x_1)e^{-sx_1}}{s} \quad (25)$$

$$\left[\frac{s}{(s^2+\alpha^2)}+\frac{(s^2-\beta_1^2+\beta_2^2)}{\beta_1^2 s}\right]\Psi(s)+\frac{sW(0)+W'(0)-\Psi(0)}{(s^2+\alpha^2)}$$
$$-\frac{s\Psi(0)+\Psi'(0)+\beta^2W(0)}{\beta_1^2 s}=\frac{\Delta\Psi(x_1,t)e^{-sx_1}-s\Delta W(x_1)e^{-sx_1}-\Delta W'(x_1)e^{-sx_1}}{(s^2+\alpha^2)}$$
$$+\frac{\Delta\Psi'(x_1)e^{-sx_1}+s\Delta\Psi(x_1)e^{-sx_1}+\beta_1^2\Delta W(x_1)e^{-sx_1}}{\beta_1^2 s} \quad (26)$$

Let $\eta_1=\frac{\beta_1^2 s}{(s^2-\beta_1^2+\beta_2^2)}+\frac{(s^2+\alpha^2)}{s}$,

$\eta_2=\frac{s}{(s^2+\alpha^2)}+\frac{(s^2-\beta_1^2+\beta_2^2)}{\beta_1^2 s}$ and Rearrange (25) and (26),

$$W(s)=C_{11}[W'(0)-\Psi(0)]+C_{12}W(0)+C_{13}\Psi'(0)+C_{14}\Psi(0)+$$
$$C_{15}[\Delta W'(x_1)-\Delta\Psi(x_1)]+C_{16}\Delta W(x_1)+C_{17}\Delta\Psi'(x_1)+C_{18}\Delta\Psi(x_1)$$

$$C_{11}=\frac{1}{s\eta_1};C_{12}=\frac{\beta_1^2}{(s^2-\beta_1^2+\beta_2^2)\eta_1}+\frac{1}{\eta_1};C_{13}=\frac{1}{(s^2-\beta_1^2+\beta_2^2)\eta_1};$$
$$C_{14}=\frac{s}{(s^2-\beta_1^2+\beta_2^2)\eta_1};C_{15}=\frac{e^{-sx_1}}{s\eta_1};$$
$$C_{16}=\frac{\beta_1^2e^{-sx_1}}{(s^2-\beta_1^2+\beta_2^2)\eta_1}+\frac{e^{-sx_1}}{\eta_1};C_{17}=\frac{e^{-sx_1}}{(s^2-\beta_1^2+\beta_2^2)\eta_1};C_{18}=\frac{se^{-sx_1}}{(s^2-\beta_1^2+\beta_2^2)\eta_1} \quad (27)$$

$$\Psi(s)=C_{21}[W'(0)-\Psi(0)]+C_{22}W(0)+C_{23}\Psi'(0)+C_{24}\Psi(0)$$
$$+C_{25}[\Delta W'(x_1)-\Delta\Psi(x_1)]+C_{26}\Delta W(x_1)+C_{27}\Delta\Psi'(x_1)+C_{28}\Delta\Psi(x_1)$$

$$C_{21} = -\frac{1}{(s^2+\alpha^2)\eta_2}; C_{22} = -\frac{s}{(s^2+\alpha^2)\eta_2} + \frac{1}{s\eta_2};$$
$$C_{23} = \frac{1}{\beta_1^2\eta_2 s}; C_{24} = \frac{1}{\beta_1^2\eta_2}; C_{25} = -\frac{e^{-sx_1}}{(s^2+\alpha^2)\eta_2}; \quad (28)$$
$$C_{26} = -\frac{se^{-sx_1}}{(s^2+\alpha^2)\eta_2} + \frac{e^{-sx_1}}{s\eta_2}; C_{27} = \frac{e^{-sx_1}}{\beta_1^2\eta_2 s}; C_{28} = \frac{e^{-sx_1}}{\beta_1^2\eta_2}$$

The modal displacements $W(x)$ and $\Psi(x)$ as shown in (29) and (30) can be obtained by applying inverse Laplace transform on both (27) and (28).

$$W(x) = S_{11}[W'(0) - \Psi(0)] + S_{12}W(0) + S_{13}\Psi'(0) + S_{14}\Psi(0) + S_{15}[\Delta W'(x_1) - \Delta\Psi(x_1)] + S_{16}\Delta W(x_1) + S_{17}\Delta\Psi'(x_1) + S_{18}\Delta\Psi(x_1) \quad (29)$$

$$\Psi(s) = S_{21}[W'(0) - \Psi(0)] + S_{22}W(0) + S_{23}\Psi'(0) + S_{24}\Psi(0) + S_{25}[\Delta W'(x_1) - \Delta\Psi(x_1)] + S_{26}\Delta W(x_1) + S_{27}\Delta\Psi'(x_1) + S_{28}\Delta\Psi(x_1) \quad (30)$$

3. Verifications

3.1 Vibration of cantilever beam attached withPZT actuator

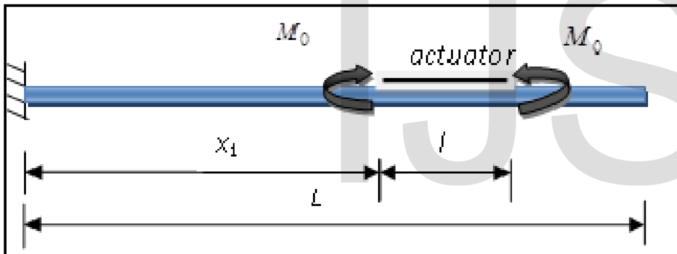

Fig.2. Cantilever beam excited by a PZT actuator

Actuator and sensors are used to analyze the natural vibrations of a beam and assess the extent of the damage by directly measuring the voltage output generated by the strained film [21], [22]. As shown in Fig.2, a PZT patch is attached to a cantilever beam as an actuator. Two concentrated moments are applied to the cantilever beam at the end of PZT patch due to a proper voltage applied to the PZT patch. In the case of harmonic vibration, the modal displacement and rotation of the cantilever beam is,

$$W(x) = S_{11}(x)[W'(0) - \Psi(0)] + S_{13}(x)\Psi'(0) + S_{17}(x)\frac{M_0}{EI} - S_{171}(x)\frac{M_0}{EI} \quad (31)$$

$$\Psi(x) = S_{21}(x)[W'(0) - \Psi(0)] + S_{23}(x)\Psi'(0) + S_{27}(x)\frac{M_0}{EI} - S_{271}(x)\frac{M_0}{EI} \quad (32)$$

Where M_0 is the moment applied by the PZT actuator; x_1 is the location of the left end of the PZT. In the expression of $S_{171}(x)$, the x_1 is replaced by $x_1 + l$, where l is the length of the actuator. There are two unknowns: $W'(0) - \Psi(0)$ and $\Psi'(0)$, which can be solved by the boundary conditions listed below,

$$W'(L) - \Psi(L) = 0, \Psi'(L) = 0 \quad (33)$$

The length of the beam is 1m with Young's modulus $4.8 \times 10^8 N/m^2$ and density $1200 kg/m^3$ .The cross section is rectangular with the dimensions $0.02 \times 0.02 m^2$. Assume $\omega = 14.606 rad/s$, $M_0 = 1N \cdot m$ and $l = 0.02L$ .The displacement using Timoshenko beam is shown in the Table 1 and Fig. 3.

TABLE 1
Displacement in Timoshenko Beam

Distance from left end	Displacement ($x = 0.1$)	Displacement ($x = 0.4$)	Displacement ($x = 0.8$)
0	-0.001728618	-0.001831169	-0.000086841
0.02	-0.001728733	-0.001834844	-0.000116655
0.04	-0.001847281	-0.001846723	-0.000196009
0.06	-0.001836841	-0.001939437	-0.000209749
0.08	-0.001944094	-0.002030995	-0.000240444
0.1	-0.001944213	-0.002291767	-0.000265752
0.12	-0.002042716	-0.002432858	-0.000347808
0.14	-0.002106311	-0.002681411	-0.000350101
0.16	-0.002193479	-0.002745829	-0.000364456
0.18	-0.002370951	-0.002853734	-0.000369311
0.2	-0.002462647	-0.002960287	-0.000427507
0.22	-0.002477219	-0.00298279	-0.000433643
0.24	-0.002697277	-0.003214136	-0.000440417
0.26	-0.002778564	-0.00326539	-0.000440897
0.28	-0.002880433	-0.003491343	-0.000443319
0.3	-0.00312659	-0.00360458	-0.000479664
0.32	-0.003245795	-0.003616963	-0.000481192
0.34	-0.003449514	-0.003730766	-0.000487013
0.36	-0.00356308	-0.00377457	-0.000500896
0.38	-0.003759847	-0.003817943	-0.000513241
0.4	-0.003950094	-0.003856863	-0.000627546
0.42	-0.004154247	-0.003930693	-0.000700271
0.44	-0.004419111	-0.003942923	-0.000700697

0.46	-0.004478018	-0.004041766	-0.000725462
0.48	-0.004877966	-0.004155032	-0.000798712
0.5	-0.004945434	-0.004184333	-0.000871751
0.52	-0.005273612	-0.004212172	-0.000901169
0.54	-0.005400918	-0.004310148	-0.000920744
0.56	-0.005673076	-0.004430299	-0.001004700
0.58	-0.005905917	-0.004571329	-0.001047801
0.6	-0.006083988	-0.004856644	-0.001100074
0.62	-0.00648443	-0.004974472	-0.001215156
0.64	-0.006619548	-0.005180038	-0.001223043
0.66	-0.00692104	-0.005294973	-0.001313301
0.68	-0.007123157	-0.005409084	-0.001361130
0.7	-0.007459306	-0.005517539	-0.001390862
0.72	-0.007670351	-0.005684769	-0.001398307
0.74	-0.008045315	-0.005729308	-0.001399341
0.76	-0.008313599	-0.005931795	-0.001511568
0.78	-0.008542906	-0.005950009	-0.001521970
0.8	-0.008884953	-0.006042841	-0.001539966
0.82	-0.009147155	-0.006206757	-0.001556834
0.84	-0.009399559	-0.006279877	-0.001583500
0.86	-0.0097322	-0.006456891	-0.001602953
0.88	-0.009951337	-0.006560958	-0.001619193
0.9	-0.01026936	-0.006698024	-0.001625730
0.92	-0.010526905	-0.007012349	-0.001708189
0.94	-0.010706729	-0.007128905	-0.001892570
0.96	-0.011093611	-0.007245989	-0.001928162
0.98	-0.011306151	-0.007359801	-0.001931410
1	-0.011663925	-0.007498321	-0.001934827

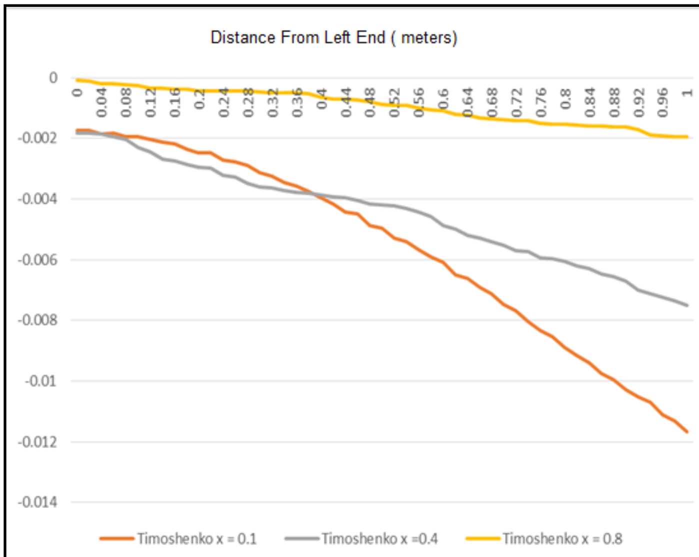

Fig. 3. Displacement of Timoshenko Beam

4. CONCLUSION

In this paper, vibration functions of a Timoshenko beam with arbitrary discontinuities and boundary conditions are derived. Heaviside's function is employed here to account for the discontinuity points in the beam so that the modal displacement and rotation can be described by a single function. Consequently the solution of vibration is significantly simplified. Various discontinuity and boundary conditions regarding to Timoshenko beam are discussed.

The application of present model to smart structure and damage detection are presented and compared with Euler-Bernoulli beam model and FEA. It turns out that present solution can give much better solution than Euler-Bernoulli beam, especially for the secondary and third natural frequency. Additionally, the deeper the crack, the more significant difference of frequencies of Timoshenko beam and Euler-Bernoulli beam.

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